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Note on a Problem in the Theory of Algebraic Manifolds.

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1. In a recent paper[1] I have solved the following problem: Let $D_1, D_2, \ldots . D_s$, be algebraic manifolds belonging to an irreducible algebraic r-dimensional variety, of which the hypersurfaces $C_1, C_2, \ldots . C_\rho$, form a base. If

$$\lambda_j D_j = \Sigma \lambda_i^j C_i ,$$

to obtain the genus of $D_1, D_2, \ldots . D_s$, complete intersection of the D's, in terms of the genera of the C's and of their mutual intersections. It was shown there that a formula answering the purpose exists, independent from the signs of the integers λ, and this formula was explicitly given (formula 15). This was finally applied to loci of spaces, or generalized scrolls, and to complete intersections in ordinary r-space.

The solution obtained was based upon Severi's formula for the genus of a manifold sum of two others[2]. However, the fundamental formula was derived only by an indirect method. It is the main object of the present note to show how the problem in question can be solved by a *complete inversion* of Severi's formula, inversion which in itself is not without interest. We will make use throughout of the notion of *virtual manifolds* considered at length in the note already referred to. With this notion the problem is purely one of algebra, and consists in the extension of a certain algebraic functional operation. The abstract problem underlying the symbolism used is considered in §§ 7, 8. The interesting result is obtained that the most general addition for-

1. The arithmetic genus of an algebraic manifold immersed in another. Annals of Mathematics, 1916. Vol. 17, p. 197.

2. Fundamenti per la geometria sulle varieta algebriche. Rendiconti del Circolo Mathematico di Palermo, 1909, p. 42.

mula analogous to Severi's is given by $C=A+B$; $\{(\alpha A+\beta)(\alpha B+\beta)\} = \{\alpha C+\beta\}$, which for $\alpha=\beta=1$ yields Severi's.

2. The symbols used will be practically those of my previous paper. It will be convenient to recall them briefly.

The genus of a manifold M will be designated by $[M]$. Let $A, B, \ldots . C$, be hypersurfaces of the fundamental irreducible algebraic variety V_r, $f(A, B, \ldots . C)$ a power series proceeding according to positive powers of the symbols $A, B, C \ldots \ldots$ Then

(a) We will denote by $[f(A, B, \ldots . C)]$ the result obtained when the constant term is left unchanged; the term in $A^a B^b \ldots . C^c$ is replaced by $[A^a B^b \ldots . C^c]$ if its degree is $\leqq r$, and by zero if it exceeds r.

(b) Let $F(x) = f(x, x, \ldots . x) = \sum_i a^i x^i$. We will write

$$\{f(A, B, \ldots . C)\} = [f(A, B, \ldots . C)] + (-1)^{r-1} \sum_{i=0}^{r-1} (-1)^i a^i.$$

This last definition is not identical with that of my paper, but is equivalent to it, as shown at the end of the introduction to it.

3. Let $D = C_1 + C_2$. Then[3]

$$\{1+D\} = \{(1+C_1)(1+C_2)\}. \tag{1}$$

The following generalizations of (1) have been obtained:

(a) If $D = \sum_{i=1}^{n} C_i$, $M = D_1 D_2 \ldots . D_h$, then

$$\{M(1+D)\} = \{M \prod_i (1+C_i)\}. \tag{2}$$

(b) If $D = C_1 - C_2$, $\{M(1+D)\} =$

$$\left\{M \frac{1+C_1}{1+C_2}\right\} \equiv \{M(1+C_1)(1+C_2)^{-1}\}, \tag{3}$$

where at the right the quotient is to be replaced by its expansion in power series.

From these formulas follows that if $D = \sum \lambda_i C_i$, where the λ's are any integers, then

$$\{M(1+D)\} = \{M \prod_i (1+C_i)^{\lambda_i}\}. \tag{4}$$

4. The formula to be proved will follow readily from the extension of (4) to the case where the λ's are any rational numbers. Suppose first that $\lambda D = C$, where λ is a positive integer. Then I propose to show that

$$\{M(1+D)\} = \{M(1+C)^{\frac{1}{\lambda}}\}, \tag{5}$$

3. The arithmetic genus. . . . formula 1. This is slightly more general than the formula originally given by Severi.

whatever the manifold M. Proceed by induction. The formula is easily verified when

$$M = M_s = A_1 A_2 \ldots . A_s, \; s > r.$$

Grant that it holds for $s > s_0$, I say that it is also true for $s = s_0$. For $\lambda M_{s_0} D = M_{s_0} C$;

$$\therefore \; \{ M_{s_0} (1+D)^\lambda \} = \{ M_{s_0} (1+C) \}. \tag{6}$$

But $M_{s_0} D^k = M_{s_0} D^{k-1}. D$.

Hence from the assumptions made follows:

$$\{ M_{s_0} D^{k-1} (1+D) \} = \{ M_{s_0} D^{k-1} (1+C)^{\frac{1}{\lambda}} \}; k > 1$$

$$\therefore \; \{ M_{s_0} D^k \} = \{ M_{s_0} D^{k-1} (1+C)^{\frac{1}{\lambda}} - 1 \}.$$

This remains true when M_{s_0} is replaced by $M_{s_0} C^h$.

$$\therefore \; \{ M_{s_0} D^k \} = \{ M_{s_0} ((1+C)^{\frac{1}{\lambda}} - 1)^k \}, k > 1. \tag{7}$$

From (6) follows

$$\{ M_{s_0} \sum_{k=0}^{\lambda} \binom{\lambda}{k} D^k \} = \{ M_{s_0} (1+C) \}.$$

This together with (7) gives finally

$$\{ M_{s_0} (1+C) \} =$$

$$\{ M_{s_0} \sum_{k=0}^{\lambda} \binom{\lambda}{k} D^k ((1+C)^{\frac{1}{\lambda}} - 1)^k + \lambda (1+D) - \lambda (1+C)^{\frac{1}{\lambda}} \} =$$

$$\{ M_{s_0} (((1+C)^{\frac{1}{\lambda}} - 1 + 1)^\lambda + \lambda (1+D) - \lambda (1+C)^{\frac{1}{\lambda}}) \}.$$

$$\therefore \; \{ M_{s_0} (1+D) - M_{s_0} (1+C)^{\frac{1}{\lambda}} \} = 0,$$

as was to be proved.

5. It remains to be shown that if $\lambda D = C_1 - C_2$, then

$$\{ M (1+D) \} = \{ M (1+C_1)^{\frac{1}{\lambda}} (1+C_2)^{-\frac{1}{\lambda}} \}. \tag{8}$$

If we set $\lambda D = D' = C_1 - C_2$, we have by (5)

$$\{ M (1+D) \} = \{ M (1+D')^{\frac{1}{\lambda}} \}. \tag{9}$$

Also,

$$\{ M(1+D') \} = \{ M(1+C_1)\,(1+C_2)^{-1} \} ;$$

$$\therefore \{ MD' \} = \{ M\,((1+C_1)\,(1+C_2)^{-1} - 1) \} .$$

Hence finally by (9)

$$\{ M(1+D) \} = \{ M(1+D')^{\frac{1}{\lambda}} \} = \{ M \sum_{k=0} \binom{\frac{1}{\lambda}}{k} . D'^{k} \} =$$

$$\{ M \sum_{k=0} \binom{\frac{1}{\lambda}}{k} D^{k} (\,(1+C_1)\,(1+C_2)^{-1} - 1)^{k} \} =$$

$$\{ M\,((1+C_1)\,(1+C_2)^{-1})^{\frac{1}{\lambda}} \} =$$

$$\{ M\,(1+C_1)^{\frac{1}{\lambda}} (1+C_2)^{-\frac{1}{\lambda}} \}$$

as was to be proved.

6. From (5) and (8) follows that if $\lambda D = C_1 \pm C_2$, then

$$\{ f(D, A, B, \ldots C) \} = \{ f(\,(1+C_1)^{\frac{1}{\lambda}} (1+C_2)^{\pm\frac{1}{\lambda}} - 1, A, B, \ldots C) \}$$

f being as usual a power series.

Let $\lambda D = \sum_{i=1}^{\rho} \lambda_i C_i$. We may set $\lambda_1 C = \pm C_1'$, $\sum_{i=2}^{\rho} \lambda_i C_i = D_1$, according as λ_1 is positive or negative. Then $\lambda D = D_1 \pm C_1' = D_1 + \lambda_1 C_1$.

$$\therefore \{ f(D, A, \ldots B) \} = \{ f(\,(1+D_1)^{\frac{1}{\lambda}} (1+C_1')^{\pm\frac{1}{\lambda}} - 1, A, \ldots B) \} ,$$

and as $\{ \Phi(C_1', A, \ldots B) \} = \{ \Phi(\,(1+C_1)^{\pm\frac{\lambda_1}{\lambda}} - 1, A, \ldots B) \}$ it follows

$$\{ f(D, A, \ldots B) \} = \{ f(\,(1+C_1)^{\frac{\lambda_1}{\lambda}} (1+D_1)^{\frac{1}{\lambda}} - 1, A, \ldots B) \}.$$

and this by a repeated application leads to the following very general formula

$$\{ f(D, A, \ldots B) \} = \{ f(\,(\prod_i (1+C_i)^{\frac{\lambda_i}{\lambda}} - 1, A, \ldots B) \} . \qquad (10)$$

From this follows immediately the fundamental formula already referred to; for if

$$\lambda_j D_j = \sum_{i=1}^{\rho} \lambda_i^j C_i, \ (j = 1, 2, \ldots . s)$$

we have

$$\{ D_1 D_2 \ldots D_s \} = \{ \prod_{i=1}^{s} (\prod_{i=1}^{\rho} (1 + C_i)^{\frac{\lambda_i^j}{\lambda_j}} - 1) \}$$

which is (15) of the paper already quoted, but in a different form. If at the right we wished, as there done, to have the sign [ ] in place of { }, we would have to add an integer N. To obtain its value in terms of the λ's we remark that according to § 2, N can be expressed as a polynomial in the ratios $\frac{\lambda_i^j}{\lambda_j}$. We may therefore make $\lambda_j = 1$, for all values of j, and find N somewhat as done *loc. cit.* § 4. This calculation will be omitted here.

7. The theory underlying the preceding discussion may be abstractly stated thus: Let $A_1, A_2, \ldots .$ be a finite or infinite set of magnitudes such that the sum $\Sigma \lambda_i A_i$, where the λ's are arbitrary integers, defines a quantity of the set. This quantity is supposed to remain unchanged when the A's are arbitrarily permuted together with their coefficients; that is, the commutative law of addition holds. Furthermore, the associative law is also assumed. Besides the above quantities, we consider their products in finite number $M = A_1 A_2 \ldots . A_s$. If r is an arbitrary but fixed positive integer, the number $r - s = d$ is called the dimensionality of M. We then define a function of M called its genus $[M]$ by the following properties:

(a) $[M]$ is uniquely determined when M is known.

(b) With the notations defined previously,

$$B = A_1 + A_2; \ \{ M \ (1 + B) \} = \{ M \ (1 + A_1) \ (1 + A_2) \} .$$

These two conditions suffice to make certain that if the numbers $[A_1^i \, A_2^j \ldots . A_s^k]$ are known we can obtain $[D_1 \, D_2 \ldots . D_h]$, where $\lambda_j D_j = \sum_{i=1}^{s} \lambda_i^j A_i$. In particular if there is a base in the

usual sense for the set of the A's, we can obtain all the genera $[M]$ as linear functions of the base genera.

Starting from the above properties, all the formulas obtained could be derived without having recourse to virtual manifolds, as becomes necessary with the more specialized geometric theory already discussed.

In this connection three questions present themselves:

(a) Is it possible to obtain other systems than that of algebraic manifolds contained in a given variety, having the properties outlined above?

(b) Given a base set $C_1, C_2, \ldots C_\rho$, the system defined by all the C's such that $\lambda C = \Sigma \lambda_i C_i$, and a system of integral values for the genera $[C^i_1 C^j_2 \ldots . C^k_\rho]$, is there an irreducible algebraic variety such that its hypersurfaces are in one-one correspondence with the system so defined?

(c) Are there laws other than the one corresponding to Severi's addition formula, and yielding results of interest?

Very little may be said at present in regard to (a) and (b), but more concerning (c), which is what we proceed to do.

8. The question must first be stated more definitely. It is proposed to consider addition formulas of the type

$$\left\{ Mf(\overline{C_1+C_2}) \right\} = \left\{ M\,\phi\,(C_1, C_2) \right\},$$

where f, ϕ, are polynomials. The simplest of this type is perhaps the following:

$$\left\{ Mf(\overline{C_1+C_2}) \right\} = \left\{ Mf(C_1)f(C_2) \right\} \tag{11}$$

which can be easily shown to lead to the result

$$\lambda D = \Sigma \lambda_i C_i\,;\ \left\{ f(D) \right\} = \left\{ \prod_i f(C_i)^{\frac{\lambda_i}{\lambda}} \right\}.$$

The difficulty with laws of this general type lies in the fact that unless f is linear, $[\overline{C_1+C_2}]$ can not be obtained readily in terms of the genera $[C^i_1\, C^k_2]$. Restricting ourselves, therefore, to the case where $[\overline{C_1+C_2}]$ can be obtained directly in terms of these genera, we propose to show that: *Addition formulas yield-*

ing directly $[M(C_1+C_2)]$ *in terms of the genera* $[MC_1^i\, C_2^k]$, *are all of the type* (11), *f being of the first degree.*

Suppose that $\{M\overline{(A+B)}\} = \{Mf(A,B)\}$. Clearly $f(A,B)$ must be symmetric in A, B, this being a direct consequence of the commutative law. Furthermore,

$$\{M(A+B+C)\} = \{M\phi(A, B, C)\},$$

where ϕ is a polynominal which, for the same reason, must be symmetric in A, B, C. Let

$$f(A, B) = \Sigma\, \alpha_{ik} A^i B^k,$$

and denote by μ the highest power at which either A or B occurs in f. Then, since

$$\phi(A, B, C) = \Sigma\, \alpha_{ik} A^i (\Sigma\, \alpha_{mn} B^m C^n)^k,$$

it is easily seen that in ϕ, A will be found at most to the power μ, while if r is sufficiently high, there will be a term containing C with the exponent μ^2.

$$\therefore\ \mu^2 = \mu,\ \mu = 1.$$

Thus the polynomial f must be of the first degree. As it is also symmetric, we have

$$f(A, B) = a(A+b)(B+b) + c,$$

where a, b, c are certain constants still in part to be determined. From this follows

$$\phi(A, B, C) = a(A+b)(a(B+b)(C+b)+c)+c,$$

and the condition of symmetry will be fulfilled only if $b = -c$, when $f(A, B) = a(A+b)(B+b) - b$.

$$\therefore\ \{a\overline{(A+B+b)}\} = \{a(A+b).a(B+b)\}.$$

If we set $a = \alpha$, $ab = \beta$, we have

$$C = A+B;\ \{M(\alpha C+\beta)\} = \{M(\alpha A+\beta)(\alpha B+\beta)\}$$

as was to be proved.

LAWRENCE, KAN., May 30, 1916

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